Graph Theory

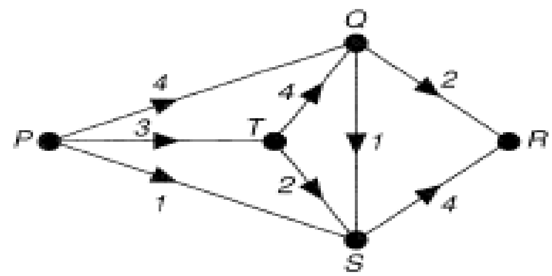

Fig. 1.13

Exercises 1

1.1[s] Write down the number of vertices, the number of edges, and the degree of each vertex, in:

(i) the graph in Fig. 1.3;
(ii) the tree in Fig. 1.14.

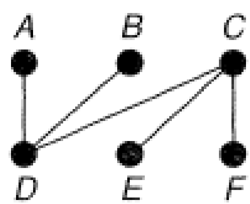

Fig. 1.14

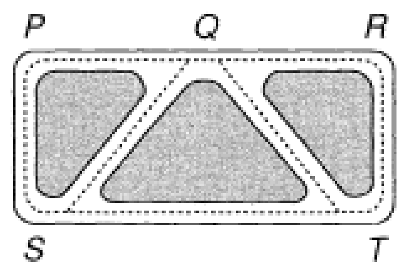

Fig. 1.15

1.2 Draw the graph representing the road system in Fig. 1.15, and write down the number of vertices, the number of edges and the degree of each vertex.

1.3[s] Figure 1.16 represents the chemical molecules of methane (CH_4) and propane (C_3H_8).

(i) Regarding these diagrams as graphs, what can you say about the vertices representing carbon atoms (**C**) and hydrogen atoms (**H**)?

(ii) There are two different chemical molecules with formula C_4H_{10}. Draw the graphs corresponding to these molecules.

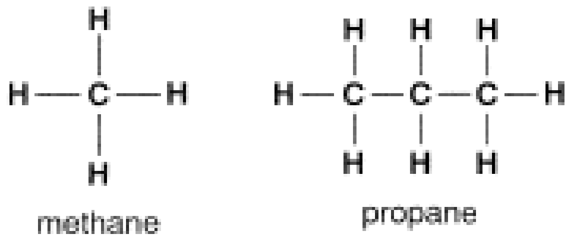

Fig. 1.16

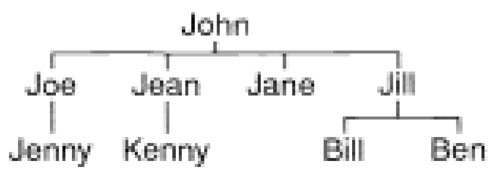

Fig. 1.17

1.4 Draw a graph corresponding to the family tree in Fig. 1.17.

1.5* Draw a graph with vertices $A, \ldots, M$ that shows the various routes one can take when tracing the Hampton Court maze in Fig. 1.18.

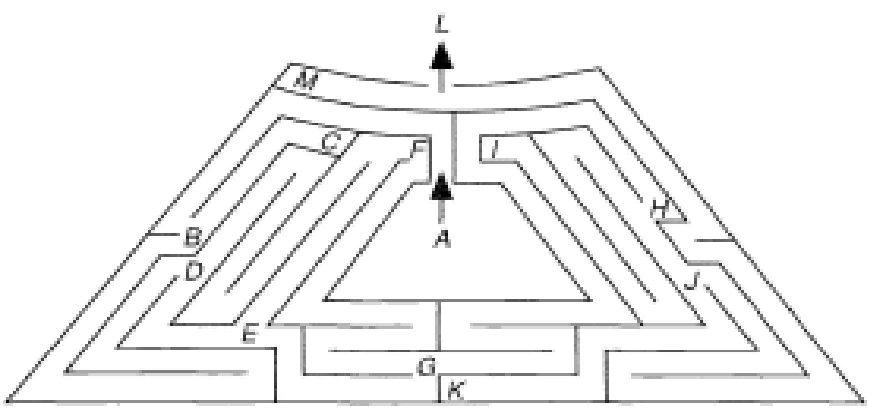

Fig. 1.18

1.6[s] John likes Joan, Jean and Jane; Joe likes Jane and Joan; Jean and Joan like each other. Draw a digraph illustrating these relationships between John, Joan, Jean, Jane and Joe.

1.7 Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders and insects. Draw a digraph representing this predatory behaviour.

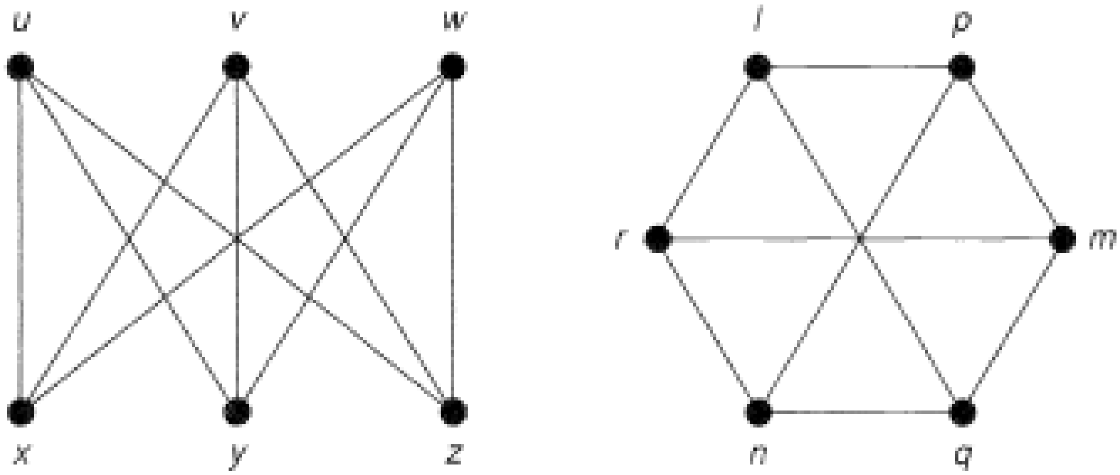

Fig. 2.3

Exercises 2

2.1[s] Write down the vertex set and edge set of each graph in Fig. 2.3.

2.2 Draw

(i) a simple graph,

(ii) a non-simple graph with no loops,

(iii) a non-simple graph with no multiple edges,

each with five vertices and eight edges.

2.3[s] (i) By suitably labelling the vertices, show that the two graphs in Fig. 2.20 are isomorphic.
(ii) Explain why the two graphs in Fig. 2.21 are not isomorphic.

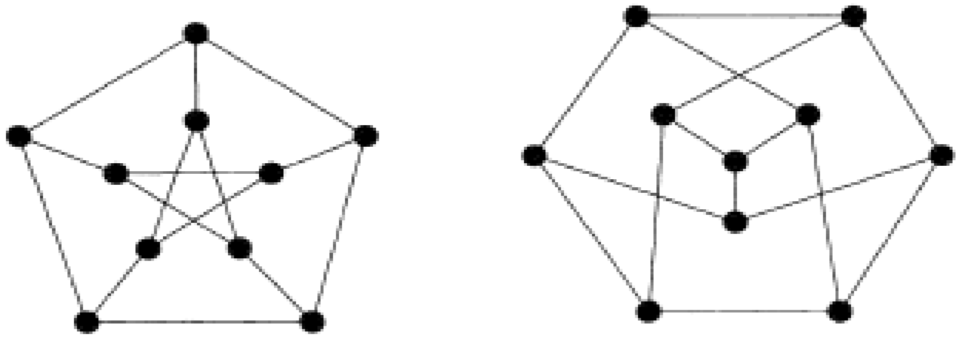

Fig. 2.20

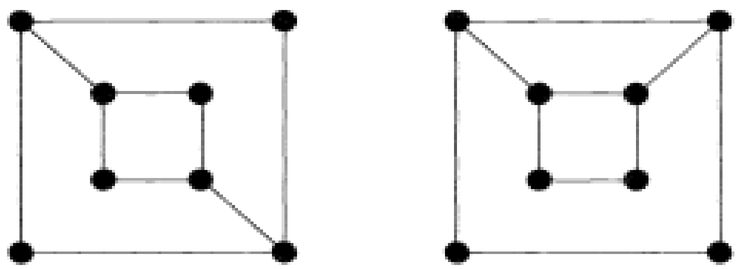

Fig. 2.21

2.4 Classify the following statements as *true* or *false:*
(i) any two isomorphic graphs have the same degree sequence;
(ii) any two graphs with the same degree sequence are isomorphic.

2.5 (i) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.
(ii) How many of these have exactly m edges?

2.6[s] Locate each of the graphs in Fig. 2.22 in the table of Fig. 2.9.

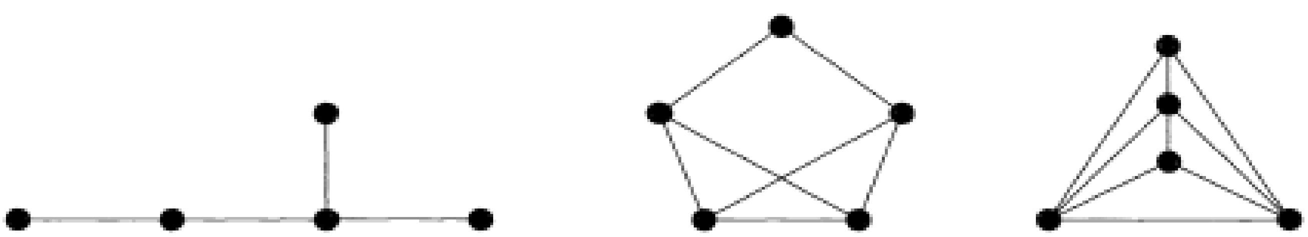

Fig. 2.22

2.7[s] Write down the degree sequence of each graph with four vertices in Fig. 2.9, and verify that the handshaking lemma holds for each graph.

2.8 (i) Draw a graph on six vertices with degree sequence (3, 3, 5, 5, 5, 5); does there exist a *simple* graph with these degrees?
(ii) How are your answers to part (i) changed if the degree sequence is (2, 3, 3, 4, 5, 5)?

2.9* If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.

2.10[s] Which graphs in Fig. 2.23 are subgraphs of those in Fig. 2.20?

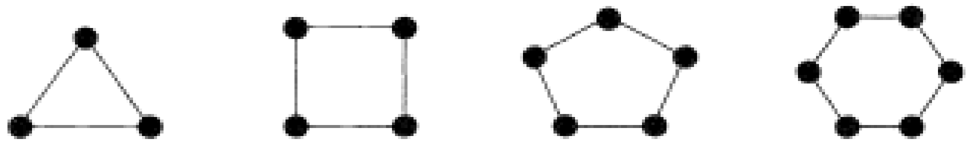

Fig. 2.23

2.11 Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k and e be an edge of G. How many vertices and edges have $G - e$, $G - v$ and $G\backslash e$?

2.12* Write down the adjacency and incidence matrices of the graph in Fig. 2.24.

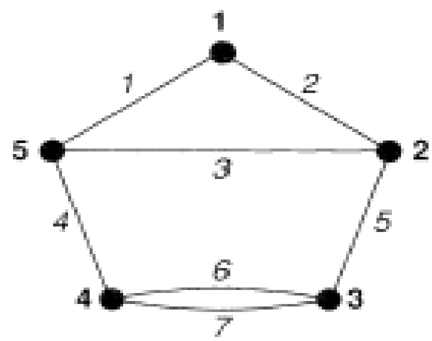

Fig. 2.24

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 2.25

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 2.26

2.13 (i) Draw the graph whose adjacency matrix is given in Fig. 2.25.
(ii) Draw the graph whose incidence matrix is given in Fig. 2.26.

2.14 If G is a graph without loops, what can you say about the sum of the entries in
(i) any row or column of the adjacency matrix of G?
(ii) any row of the incidence matrix of G?
(iii) any column of the incidence matrix of G?

2.15* If G is a simple graph with edge-set $E(G)$, the **vector space of G** is the vector space over the field $\mathbf{Z}_2$ of integers modulo 2, whose elements are subsets of $E(G)$. The sum $E + F$ of two subsets E and F is the set of edges in E or F but not both, and scalar multiplication is defined by $1.E = E$ and $0.E = \varnothing$. Show that this defines a vector space over $\mathbf{Z}_2$, and find a basis for it.

Exercises 3

3.1[s] Draw the following graphs:

(i) the null graph N_5;
(ii) the complete graph K_6;
(iii) the complete bipartite graph $K_{2,4}$;
(iv) the union of $K_{1,3}$ and W_4;
(v) the complement of the cycle graph C_4.

3.2[s] How many edges has each of the following graphs:
(i) K_{10}; (ii) $K_{5,7}$; (iii) Q_4; (iv) W_8; (v) the Petersen graph?

3.3 How many vertices and edges has each of the Platonic graphs?

3.4[s] In the table of Fig. 2.9, locate all the regular graphs and the bipartite graphs.

3.5 Give an example (if it exists) of each of the following:

(i) a bipartite graph that is regular of degree 5;
(ii) a bipartite Platonic graph;
(iii) a complete graph that is a wheel;
(iv) a cubic graph with 11 vertices;
(v) a graph (other than K_4, $K_{4,4}$ or Q_4) that is regular of degree 4.

3.6[s] Draw all the simple cubic graphs with at most 8 vertices.

3.7 The **complete tripartite graph** $K_{r,s,t}$ consists of three sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw the graphs $K_{2,2,2}$ and $K_{3,3,2}$ and find the number of edges of $K_{3,4,5}$.

3.8 A simple graph that is isomorphic to its complement is **self-complementary**.

(i) Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer.
(ii) Find all self-complementary graphs with 4 and 5 vertices.
(iii) Find a self-complementary graph with 8 vertices.

3.9* The **line graph** $L(G)$ of a simple graph G is the graph whose vertices are in one–one correspondence with the *edges* of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.

(i) Show that K_3 and $K_{1,3}$ have the same line graph.
(ii) Show that the line graph of the tetrahedron graph is the octahedron graph.
(iii) Prove that, if G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.
(iv) Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.
(v) Show that $L(K_5)$ is the complement of the Petersen graph.

3.10* An **automorphism** φ of a simple graph G is a one–one mapping of the vertex set of G onto itself with the property that $\varphi(v)$ and $\varphi(w)$ are adjacent whenever v and w are. The **automorphism group** $\Gamma(G)$ of G is the group of automorphisms of G under composition.

(i) Prove that the groups $\Gamma(G)$ and $\Gamma(\bar{G})$ are isomorphic.

(ii) Find the groups $\Gamma(K_n)$, $\Gamma(K_{r,s})$ and $\Gamma(C_n)$.

(iii) Use the results of parts (i) and (ii) and Exercise 3.9(v) to find the automorphism group of the Petersen graph.

Exercises 4

4.1[s] Find another solution of the eight circles problem.

4.2[s] Show that there is a gathering of five people in which there are no three people who all know each other and no three people none of whom knows either of the other two.

4.3[s] Find a solution of the four cubes problem for the set of cubes in Fig. 4.12.

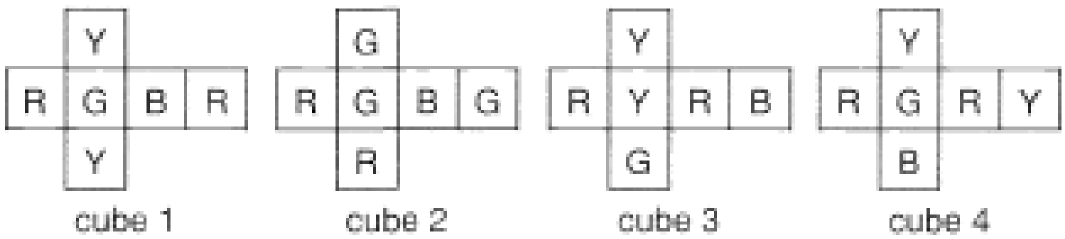

Fig. 4.12

4.4 Show that the four cubes problem in Fig. 4.13 has no solution.

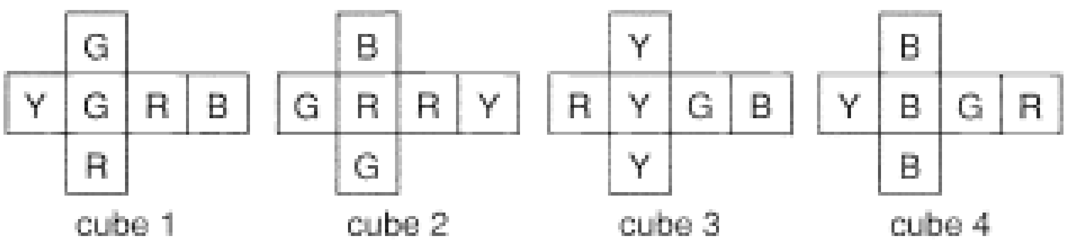

Fig. 4.13

4.5* Prove that the solution of the four cubes problem in the text is the only solution for that set of cubes.

Exercises 5

5.1[s] In the Petersen graph, find
(i) a trail of length 5;
(ii) a path of length 9;
(iii) cycles of lengths 5, 6, 8 and 9;
(iv) cutsets with 3, 4 and 5 edges.

5.2[s] The **girth** of a graph is the length of its shortest cycle. Write down the girths of (i) K_9; (ii) $K_{5,7}$; (iii) C_8; (iv) W_8; (v) Q_5; (vi) the Petersen graph; (vii) the graph of the dodecahedron.

5.3 Prove the converse of Theorem 5.1 – that if each cycle of a graph G has even length, then G is bipartite.

5.4[s] Prove that a simple graph and its complement cannot both be disconnected.

5.5[s] Write down $\kappa(G)$ and $\lambda(G)$ for each of the following graphs G:
(i) C_6; (ii) W_6; (iii) $K_{4,7}$; (iv) Q_4.

5.6 (i) Show that, if G is a connected graph with minimum degree k, then $\lambda(G) \leq k$.
(ii) Draw a graph G with minimum degree k for which $\kappa(G) < \lambda(G) < k$.

5.7 (i) Prove that a graph is 2-connected if and only if each pair of vertices are contained in a common cycle.
(ii) Write down a corresponding statement for a 2-edge-connected graph.

5.8 Let G be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, m edges and t triangles.
(i) Given that $\mathbf{A}$ is the adjacency matrix of G, prove that the number of walks of length 2 from v_i to v_j is the ij-th entry of the matrix $\mathbf{A}^2$.
(ii) Deduce that $2m$ = the sum of the diagonal entries of $\mathbf{A}^2$.
(iii) Obtain a result for the number of walks of length 3 from v_i to v_j, and deduce that $6t$ = the sum of the diagonal entries of $\mathbf{A}^3$.

5.9 In a connected graph, the **distance** $d(v,w)$ from v to w is the length of the shortest path from v to w.
(i) If $d(v, w) \geq 2$, show that there exists a vertex z such that $d(v, z) + d(z, w) = d(v, w)$.
(ii) In the Petersen graph, show that $d(v, w) = 1$ or 2, for any distinct vertices v and w.

5.10* Let G be a simple graph on $2k$ vertices containing no triangles. Show, by induction on k, that G has at most k^2 edges, and give an example of a graph for which this upper bound is achieved. (This result is often called *Turán's extremal theorem.*)

5.11* (i) Prove that, if two distinct cycles of a graph G each contain an edge e, then G has a cycle that does not contain e.
(ii) Prove a similar result with 'cycle' replaced by 'cutset'.

5.12* (i) Prove that, if C is a cycle and C^* is a cutset of a connected graph G, then C and C^* have an even number of edges in common.
(ii) Prove that, if S is any set of edges of G with an even number of edges in common with each cutset of G, then S can be split into edge-disjoint cycles.

5.13* A set E of edges of a graph G is **independent** if E contains no cycle of G. Prove that:

(i) any subset of an independent set is independent;

(ii) if I and J are independent sets of edges with $|J| > |I|$, then there is an edge e that lies in J but not in I with the property that $I \cup \{e\}$ is independent.

Show also that (i) and (ii) still hold if we replace the word 'cycle' by cutset'.